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THE EFFECTIVE NUMBER OF PULSES PER BEAMWIDTH FOR A SCANNING RAD--ETC(U)

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NRL MEMORANDUM REPORT No. 34

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L. V. Blake

RADIO DIVISION II

8 July 1952

NAVAL RESEARCH LABORATORY, WASHINGTON, D.C.

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By

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8 July 1952

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ABSTRACT

From the viewpoint of minimum-detectable-signal or radar-maximum-range theory, the number of pulses received from a target during one scan of the radar antenna is an important quantity. This has usually been arbitrarily taken to be the number occurring between half-power points of the beam. A mathematical analysis of the "integration" effect for the train of pulses of varying amplitude received when the antenna beam shape is Gaussian shows that optimum results are obtained when the integration is performed over an angle equal to about 0.84 times the half-power (one-way) beamwidth. The signal-to-noise ratio obtained by this integration is equivalent to that of a rectangular-shaped beam of 0.47 times the half-power width of a Gaussian-shaped beam. Thus the number of pulses received is 0.47 times the number usually assumed. This corresponds to a reduction in calculated system sensitivity of about 1.6 db.

PROBLEM STATUS

The work described in this report is an independent part of a larger, more general problem. The parent problem is a continuing one on which additional work will be done as new ideas or specific requirements arise.

AUTHORIZATION

NRL Problem R02-50
NR 502-500

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THE EFFECTIVE NUMBER OF PULSES PER BEAMWIDTH FOR A SCANNING RADAR

INTRODUCTION

When a scanning radar of beamwidth β, angular scanning speed ω, and pulse repetition frequency F scans past a point target, it is customary to assume that a train of n pulses is received from the target, and that $n = \beta F/\omega$. This assumption is based on idealization of the antenna beam pattern -- that is, it is assumed that the beam has uniform gain over an angle β, and zero gain elsewhere. In the real case, the gain is variable and a reasonable representation of the variation is the Gaussian or "error" function. The beamwidth is defined as the width of this function between half-power points, or 0.707 voltage points. (If the power pattern is assumed to be Gaussian, then the voltage pattern is also of Gaussian form, the only change being in the coefficient of the exponent.)

The assumed "number of pulses" figures in the calculation of system sensitivity and range of detection -- specifically, it affects the calculation of the minimum detectable signal level, because of the effect of integration.[1] Briefly this means that the "received signal" consists of the net effect of the train of pulses. The net effect is analyzed in terms of the signal-to-noise ratio for the n integrated pulses as compared with observation of a single pulse. In general the resultant signal-to-noise ratio increases with number of uniform-amplitude pulses integrated (though not necessarily linearly). It is important to know what number of pulses to use in computing the effective signal-to-noise ratio for the train of pulses received as an antenna scans past a target.

The procedure of taking the number between half-power points of the beamwidth is obviously an arbitrary one. Pulses of reduced amplitude are received far out from the beam center, the amplitude varying with angle, θ, according to some function which is here assumed to be of the form $e^{-k\theta^2}$. The two questions to be answered are: (1) how many of these pulses are actually integrated -- i.e., how far out on the edges of the beam do the pulses actually contribute to the "signal" observed by the radar operator; (2) what is the signal-to-noise ratio resulting from

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[1] If this is not a familiar concept, an excellent discussion of it may be found in Volume I of the M.I.T. Radiation Laboratory Series, "Radar System Engineering," pp 41-7; Section 2.11 ("Effect of Storage on Radar Performance").

this integration of many pulses of different amplitudes, and what "rectangular"-shaped beamwidth would give this same signal-to-noise ratio after integration? If these questions were answered, it would then be possible, for minimum-detectable-signal and maximum-range computations, to take as the equivalent number of full-amplitude pulses integrated, n, the number occurring in this equivalent rectangular beamwidth.

SUMMARY OF RESULTS

On the basis of some reasonable assumptions concerning the nature of the integration process, it has been found possible to deduce such an equivalence. The result obtained is that the equivalent rectangular beamwidth is 0.473 times the half-power width of a Gaussian-shaped beam. Hence the equivalent number of full-amplitude pulses integrated is 0.473 times the number usually assumed. On the basis that minimum detectable signal power varies inversely as the square root of the number of pulses integrated, the system sensitivity computed on this new basis compared with the former assumption (as to number of pulses integrated) is smaller by the factor $\sqrt{.473} = 0.69$ (equivalent power ratio). This is about 1.6 db. Thus a "correction" for this effect may be applied to computations of system sensitivity already made on the previous basis (a 1.6 db correction).

The answer obtained to the first of the two questions -- namely, how far out from the beam center can the integration process be carried profitably -- is 0.84 times the distance out to the half-power point. What this means is that integration out to this point gives an improvement in signal-to-noise ratio greater than the improvement obtained by integrating over lesser or greater portions of the beam. There is, in other words, an optimum "integration angle." The explanation of this phenomenon in general terms is that in any integration process the noise as well as the signal is being integrated, and the process is profitable only so long as it favors the build-up of signal compared to the build-up of noise.[2] Inasmuch as the signal amplitude is getting smaller and smaller with angular distance from the beam center, while the noise is remaining constant in amplitude, it is easy to see why the integration process should not be carried too far. On the other hand, it is probably also obvious that, near the beam center where the slope of signal amplitude vs angle is small, integration will be beneficial. Obviously then there is somewhere an optimum stopping point, which our analysis has shown to be .844 of the distance from the beam center to the half-power point.

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2 For explanation of why the two build up differently so as to favor the signal when the pulses are of constant amplitude see the reference of footnote 1.

In going from this result to an "equivalent rectangular beamwidth" it is of course assumed that the observer somehow has the ability to integrate only over this optimum angle. Possibly this is not too rash an assumption. One can at least imagine than an observer automatically varies his "observation angle" over a reasonable range, in an attempt to detect signals, and that this range would include the optimum angle. At any rate, it is necessary to make the assumption in order to make a mathematical analysis. Moreover, the result obtained is, if not correct, on the optimistic side, meaning that the correction factor may be greater than 1.6 db, but not less (insofar as this particular possible source of error is concerned).

MATHEMATICAL FORMULATION AND SOLUTION OF THE PROBLEM

The one-way antenna voltage gain pattern for an assumed Gaussian* beam shape is:

(1) $$G_1(\theta) = e^{-.347(\theta/\theta_1)^2}$$

where θ is angle measured with respect to the beam center, θ_1 is the coordinate of the half-power (.707 voltage) point, and the gain is normalized to unity at mid-beam. (The "half-power beamwidth" is $2\theta_1$).

For 2-way (radar) propagation this becomes

(2) $$G_2(\theta) = G_1^2(\theta) = e^{-.694(\theta/\theta_1)^2}$$

The integration process we are concerned with occurs after detection (demodulation). Since we are concerned with small signals in any discussion of minimum levels of discernibility, the effect of the detector on low-signal levels must be taken into account. This is approximately a square-law effect (even for a so-called linear detector), and so the law of variation of signals as a function of position in the beam, as finally observed in the receiver output, is

(3) $$G_3(\theta) = G_2^2(\theta) = e^{-1.387(\theta/\theta_1)^2}$$

The effect of the integration process is analyzed in terms of its effect on signal-to-noise ratio. After detection (demodulation),

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* The representation of antenna beam shape by the Gaussian function is based on empirical studies of practical antenna beams. This representation is typically valid over at least the range of beam arc of interest here.

the measure of signal-to-noise voltage ratio may be taken to be[3]

$$(4) \qquad R = \frac{\overline{E}_{S+N} - \overline{E}_N}{\sigma_N}$$

where $\overline{E}_{S+N}$ is the average value of signal-plus-noise, $\overline{E}_N$ is the average value of the noise in the absence of a signal, and σ_N is the root-mean-square deviation from the mean (standard deviation) of the fluctuations of noise. In this expression we may call $(\overline{E}_{S+N} - \overline{E}_N)$ the "signal."

By an integration process it is meant that the effect of successive signals (and noise) is additive. It is a well known theorem of mathematical probability theory that the average value of the sum of n random variables is equal to the sum of the average values. Hence for the integration of n pulses all having the same average value the numerator of (4) is multiplied by n. On the other hand, the standard deviation of the sum of n random variables is equal to the square root of the sum of the squares of the individual standard deviations. Hence for the integration of n noise voltages the denominator of (4) is increased by the square root of n. This leads to the well-known result that R is increased by the square root of n; this is the improvement in video signal-to-noise voltage ratio obtained by video integration of n pulses all having the same average value. Because of the square-law relation between small values of pre-detection and post-detection signal-to-noise ratios, the equivalent pre-detection improvement factor is $n^{1/4}$. This refers to voltage signal-to-noise ratio. The equivalent pre-detection signal-to-noise power ratio improvement is therefore $n^{1/2}$. This is the form in which experimental work confirming this result has usually been expressed.

When the integrated pulses are varying in amplitude in accordance with the beam pattern, the "signal" $S = \overline{E}_{S+N} - \overline{E}_N$ is a variable, following the law of equation (3). (It is the quantity S which varies as the square of the pre-detection signal level, for signals of the order of the noise or smaller.) That is,

$$S(\theta) = S(0)\, e^{-1.387(\theta/\theta_1)^2}$$

---

3 This concept of post-detection signal-to-noise ratio was proposed by the writer in an appendix to NRL report R-3123, and subsequently also in a book, "Threshold Signals," by Lawson and Uhlenbeck (McGraw-Hill, 1950; pp. 161-163).

Designating S(0) as S_o, and assuming a high "density" of pulses, the result of integrating from an angle $-\theta_2$ to $+\theta_2$ is:

$$R = \frac{k\int_{-\theta_2}^{\theta_2} S_o e^{-1.387(\theta/\theta_1)^2} d\theta}{\left[k\int_{-\theta_2}^{\theta_2} \sigma_N^2 d\theta\right]^{1/2}} \tag{6}$$

$$= R_o\sqrt{\frac{2k}{\theta_2}}\int_0^{\theta_2} e^{-1.387(\theta/\theta_1)^2} d\theta$$

where $R_o = S_o/\sigma_N$. Here k is the pulse density, in pulses per unit angle. That is, if the pulse rate is F and the scanning speed ω, $k = F/\omega$. The number of pulses integrated in the interval $-\theta_2$ to $+\theta_2$ is of course $2k\theta_2$.

(The assumption of high pulse density is necessary to justify the use of an integral representation of the sum. In Appendix A, however, it is shown that this representation also gives valid results for low-pulse-density cases. It represents, for such cases, the statistical average result, which is in fact the only generally meaningful and useful result that could be obtained.)

Therefore the improvement relative to the mid-beam single-pulse signal-to-noise ratio is

$$A = R/R_o = \sqrt{\frac{2k}{\theta_2}}\int_0^{\theta_2} e^{-1.387(\theta/\theta_1)^2} d\theta \tag{7}$$

The object of this analysis is to discover, if possible, the value of θ_2 which results in a maximum value of A. Mathematically this corresponds to the condition

$$\frac{dA}{d\theta_2} = 0 \tag{8}$$

Physical reasoning indicates that if a solution to (8) exists it will correspond to a maximum value of A, and not a minimum, and that there will be only one maximum.

From (7) we get

$$\frac{dA}{d\theta} = \sqrt{\frac{2k}{\theta_2}}\, e^{-1.387(\theta_2/\theta_1)^2} - \frac{1}{\theta_2^{3/2}}\sqrt{\frac{k}{2}}\int_0^{\theta_2} e^{-1.387(\theta/\theta_1)^2} d\theta \tag{9}$$

and setting the right side equal to zero gives

$$(10) \qquad 2\theta_2 e^{-1.387(\theta_2/\theta_1)^2} = \int_0^{\theta_2} e^{-1.387(\theta/\theta_1)^2} d\theta$$

It will be noted that the k has now dropped out, indicating that the value of θ_2 giving maximum A does not depend on k, although the assumption that $k\theta_1$ is large still applies.

The integral on the right is of the form of the well known error function which is tabulated, and hence a graphical solution to (10) may be obtained. This yields

$$(11) \qquad \theta_2 = 0.844\,\theta_1$$

as the optimum value.

The next step is to calculate the "equivalent rectangular beamwidth." For a rectangular beam, of width 2Φ, the integration improvement factor would be $\sqrt{2k\Phi}$ (the number of pulses integrated would be $2k\Phi$.) We therefore write (from (7))

$$(12) \qquad \sqrt{2k\Phi} = \sqrt{\frac{2k}{\theta_2}} \int_0^{\theta_2} e^{-1.387(\theta/\theta_1)^2} d\theta$$

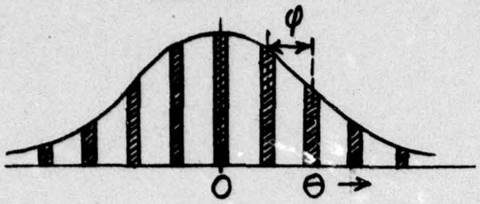
Putting in the value $\theta_2 = 0.84\theta_1$ and solving for Φ gives $2\Phi = 0.473(2\theta_1)$ as the equivalent rectangular beamwidth. This represents a reduction of radar sensitivity of 1.6 db compared to the assumption of a rectangular beam of width $2\theta_1$.

FURTHER ANALYSIS OF LOW-PULSE-DENSITY CASE

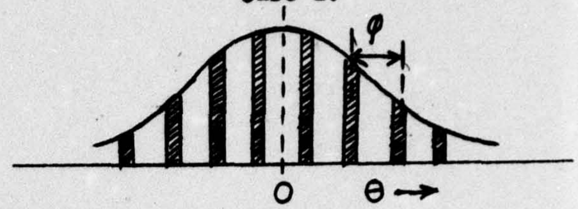
As has been pointed out, the foregoing results apply statistically to low-pulse-density as well as high-pulse-density cases. When the analysis was first made, it was suspected that this was the case, but the rigorous proof of the fact, given in Appendix A, was not made until later. Therefore a detailed analysis of the low-pulse-density case was attempted. The results obtained may be of interest because they indicate what may be expected during any one scan of a low-pulse-density radar.

For low pulse densities it is necessary to use summations. Moreover, it is necessary to assume some particular positioning of the pulses within the beam. Actually all possible positions will occur. Two possible ones which are symmetrical were assumed, as indicated in the following diagrams:

Case 1.



Case 2.



That is, in Case 1, one of the pulses is assumed to fall at the exact beam center. If the angular spacing of the pulses is ϕ, then the other pulses fall at $\theta = \pm\phi, \pm 2\phi, \pm 3\phi \ldots$ etc. In Case 2, there is no pulse at the center but the pulses are symmetrically positioned with respect to the center -- that is, they fall at $\theta = \pm \phi/2, \pm 3\phi/2, \pm 5\phi/2 \ldots\ldots\ldots\ldots$ etc.

The expressions corresponding to (7) for these two cases are

Case 1.

(13a)
$$A_1 = \frac{1 + 2\sum_{m=1}^{\left(\frac{n-1}{2}\right)} e^{-1.387\left(\frac{m\phi}{\theta_1}\right)^2}}{\sqrt{n}}$$

Case 2.

(13b)
$$A_2 = \frac{2\sum_{m=1}^{n/2} e^{-1.387\left[\frac{(2m-1)\phi}{2\theta_1}\right]^2}}{\sqrt{n}}$$

where n is the number of pulses integrated.

Expressions of this kind cannot be differentiated with respect to n unless the summations can somehow be put into closed form. This does not appear to be directly possible, in terms of any known functions. However, it is possible if the Gaussian function is approximated, over the range of values of interest, by a second degree parabola. This requires a suitable choice of the coefficients a_1, a_2, a_3:

(14)
$$e^{-1.387x^2} = a_1 x^2 + a_2 x + a_3$$

Choosing these coefficients by the method of least squares[4] gives

$$a_1 = -.54$$
$$a_2 = -.28$$
$$a_3 = 1$$

A comparison of this approximation with the actual function is shown in the curves of Figure 1.

Substituting this approximation in (13a) gives

$$A_1(n) = n^{-1/2}\left[1 + 2\sum_{m=1}^{\left(\frac{n-1}{2}\right)} 1 \quad - 2\left(\frac{.28\phi}{\theta_1}\right)\sum_{m=1}^{\left(\frac{n-1}{2}\right)} m - 2\left(\frac{.54\phi^2}{\theta_1^{\,2}}\right)\sum_{m=1}^{\left(\frac{n-1}{2}\right)} m^2\right] \tag{15}$$

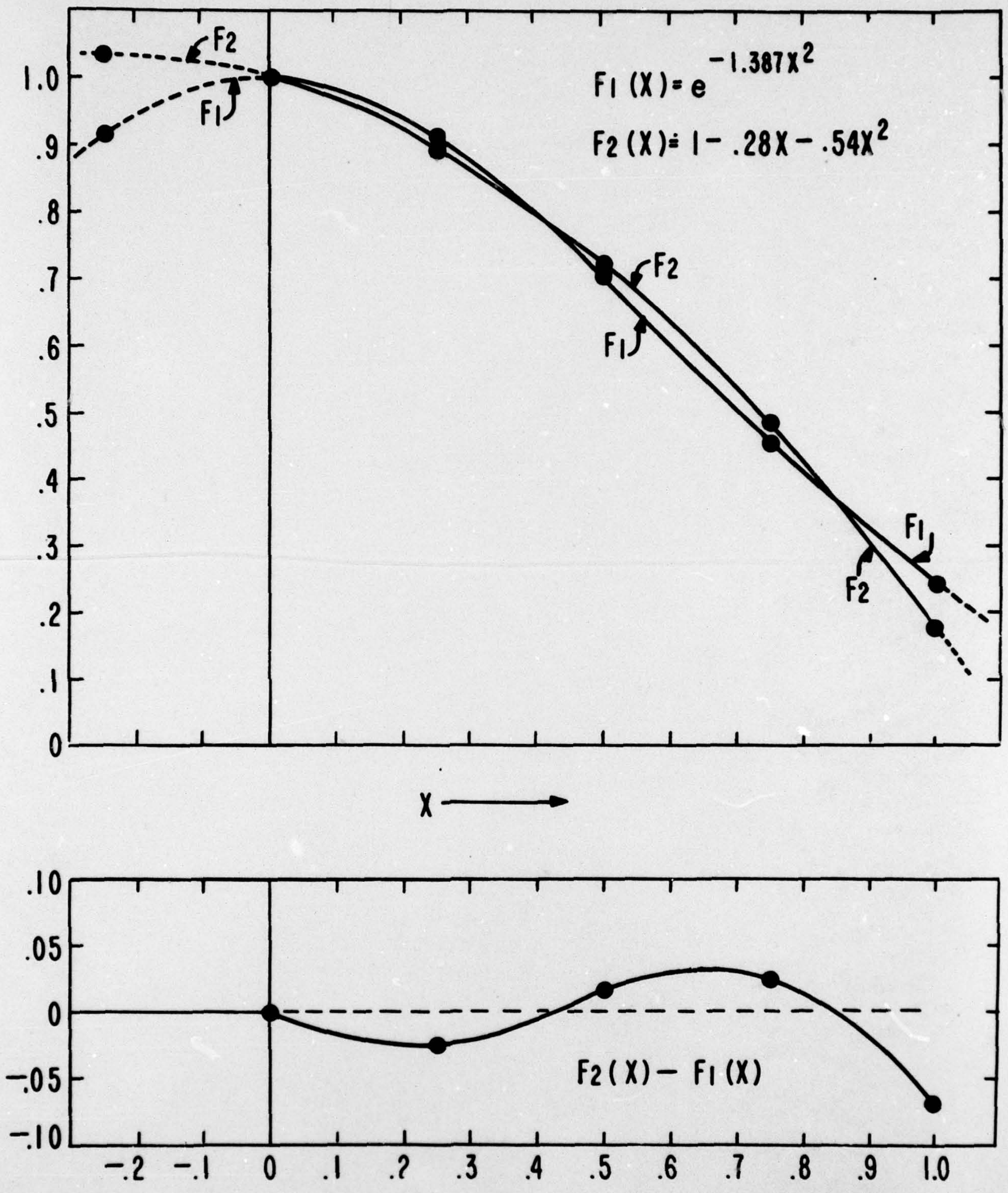
The first two of these summations are readily evaluated, as follows:

$$\sum_{m=1}^{\left(\frac{n-1}{2}\right)} 1 = \frac{n-1}{2} \tag{16}$$

$$\sum_{m=1}^{\left(\frac{n-1}{2}\right)} m = \frac{n^2-1}{2} \qquad \text{(arithmetical progression)} \tag{17}$$

---

4 This was done by expanding $e^{-1.387x^2}$ in the first three terms of a series of Legendre polynomials, which can be shown to give a result satisfying the "least squares" criterion. I am indebted to Mr. L.G. McCracken for suggesting this method. The value $a_3 = 1$ may seem somewhat improbable as a least-square-fit value. It was obtained by slide-rule computation and is therefore not exact, but a more accurate computation was deemed unnecessary because of the good results obtained with these values of the coefficients.



$F_1(X) = e^{-1.387X^2}$
$F_2(X) = 1 - .28X - .54X^2$
F2
F1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
X
.10
.05
−.05
−.10
F2(X) − F1(X)
−.2
−.1
.1
.2
.3
.4
.5
.6
.7
.8
.9

FIG. 1

The third summation can be shown[5] to be

$$(18) \qquad \sum_{m=1}^{\left(\frac{n-1}{2}\right)} m^2 = \frac{n(n^2-1)}{24}$$

Thus (15) can be expressed in completely closed form, giving, for Case 1:

$$(19) \qquad A_1(n) = n^{-1/2}\left[-.045\left(\frac{\varphi}{\theta_1}\right)^2 n^3 - .07\left(\frac{\varphi}{\theta_1}\right)n^2 + \left\{1 + .045\left(\frac{\varphi}{\theta_1}\right)^2\right\}n + .07\frac{\varphi}{\theta_1}\right]$$

Differentiating this with respect to n and setting the result equal to zero gives

$$(20) \qquad n^3 + 0.935\left(\frac{\theta_1}{\varphi}\right)n^2 - \left[4.44\left(\frac{\theta_1}{\varphi}\right)^2 + 0.2\right]n + 0.311\left(\frac{\theta_1}{\varphi}\right) = 0$$

A similar process applied to (13b) gives for Case 2:

$$(21) \qquad A_2(n) = n^{-1/2}\left[-.045\left(\frac{\varphi}{\theta_1}\right)^2 n^3 - .07\left(\frac{\varphi}{\theta_1}\right)n^2 + \left\{1 + .045\left(\frac{\varphi}{\theta_1}\right)^2\right\}n\right]$$

and as the relation for maximum R_2:

$$(22) \qquad n^2 + 0.935\left(\frac{\theta_1}{\varphi}\right)n - \left[4.44\left(\frac{\theta_1}{\varphi}\right)^2 + 0.2\right] = 0$$

---

[5] This result was given to me by Mr. W.S. Alderson, who obtained it by application of a generating-function technique to a difference equation. The general form of his result is $\sum_{m=1}^{N} m^2 = \frac{N(N+1)(2N+1)}{6}$

By multiplying (20) through by n^{-1} a striking similarity to (22) is observed, which indicates that the optimum value of n for the two cases is different only when the optimum value is small -- i.e., when the pulse density is quite low.

Equations (19) through (22) were applied to a particular case representing typical radar operation at low pulse density. The values assumed were: scanning speed (ω) 200 degrees/second, beam-width ($2\theta_1 = \beta$) 3 degrees, and pulse rate (F) 300 per second. Hence, $\varphi/\theta_1 = 2\omega/\beta F = 0.444$. (These figures are reasonable practical values, but so far as is known they do not represent any actual radar system.) For Case 1 the optimum value found was n = 3.8, and for Case 2, n = 3.83. (For Case 1 the solution of the cubic equation (20) was obtained graphically, and only two significant figures were considered justified. For Case 2 equation (22) is of course quadratic and the solution was found to three significant figures. The two results are obviously very nearly the same.)

In Case 1, the actual number of pulses must be an odd integer, and thus would be either n = 3 or n = 5. To determine which of these values to take, A_1 was calculated, from (19), as follows:

n	3	3.8	5
$A_1(n)$	1.47	1.50	1.43

Thus n = 3 is the best value.

For Case 2 the number must be an even integer, which would obviously be n = 4. Substituting this value in (21) gives $A_2 = 1.49$. The consistency of all these results seems very good.

The expressions (19) through (22) obtained for low pulse density are in no way restricted to low-density cases, although (11) may give a more accurate result in high-pulse-density cases because of the approximation (14) used in obtaining equations (19) through (22). When high pulse density is assumed, for equations (20) and (22) it is apparent that $n \gg 1$, $\varphi/\theta_1 \ll 1$, and $n\varphi/\theta_1$ is of the order of 1. Applying these relations to both (20) and (22) so as to discard terms involving $n^p(\varphi/\theta_1)^q$ where $q > p$, both equations reduce to a form involving the variable $u = \dfrac{n\varphi}{\theta_1}$:

$$u^2 + 0.935u - 4.44 = 0 \qquad (23)$$

for which the solution is u = 1.698

In the symbols of equations (6) through (12) this means:

$$\theta_2 = 0.849\theta_1 \tag{24}$$

Comparing this with (11) it is seen that the two results are in very good agreement. This indicates that the approximation (14) is a good one.

ACKNOWLEDGMENT

The need for a solution to this problem was pointed out to me by Mr. A. A. Varela. I am also indebted to Mr. Varela for helpful discussions during my work on the problem.

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APPENDIX A

This Appendix is a proof that the integral solution of the problem is valid not only for a dense-pulse case but also for low-pulse-density cases. This proof was worked out by Mr. W. S. Alderson.

Suppose that the total angular interval from $-\theta_2$ to $+\theta_2$ contains $(n + 1)$ pulses separated by angular intervals of width $\Delta\theta$. Within the range $-\theta_2$ to $+\theta_2$ the set of n pulses may take an infinite number of positionings (that is, any one of the pulses may have a position anywhere within the interval $\Delta\theta$). On any one scan of the radar a particular positioning will occur, and a particular value of R will be observed (see equations 4, 6, and 13). For purposes of computing radar sensitivity or maximum range, the average value of R over a great many observations (or scans) is required.

On any single scan, let the pulses occurring in the interval $-\theta_2$ to $+\theta_2$ be numbered from $-n/2$ to $+n/2$. Let the angular position of the j^{th} pulse be α_j. Using notation similar to that of equation 6, the amplitude of the j^{th} signal pulse after detection may be denoted as $S(\alpha_j)$. Then the integrated signal-to-noise ratio after detection, as given for a low-pulse-density case by equation 13, will be

$$R = \frac{\sum_{j=-n/2}^{n/2} S(\alpha_j)}{\sqrt{n+1}\ \sigma}$$

where σ is the rms noise voltage.

This result holds for any particular positioning of the pulses. The average over all possible positionings may be written

$$\overline{R} = \frac{1}{\sqrt{n+1}\ \sigma} \overline{\sum_{j=-n/2}^{n/2} S(\alpha_j)}$$

If one interval $\Delta\theta$ is sub-divided into N equally spaced sub-intervals, so that the m^{th} sub-interval of the j^{th} interval is located at angular position α_{jm}, we may write

$$\overline{\sum_{j=-n/2}^{n/2} S(\alpha_j)} = \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \sum_{j=-n/2}^{n/2} S(\alpha_{jm})$$

$$= \sum_{j=-n/2}^{n/2} \left[\frac{1}{\Delta\theta} \int_{(j-\frac{1}{2})\Delta\theta}^{(j+\frac{1}{2})\Delta\theta} S(\theta)\, d\theta \right]$$

$$= \frac{1}{\Delta\theta} \int_{-\theta_2}^{\theta_2} S(\theta)\, d\theta$$

Hence

$$\overline{R} = \frac{\frac{1}{\Delta\theta} \int_{-\theta_2}^{\theta_2} S(\theta)\, d\theta}{\sqrt{n+1}\ \sigma}$$

Here $1/\Delta\theta$ corresponds to the "pulse density," k, in equation 6. Similarly the quantity $(n+1)$ can be re-written as $(2\ k\ \theta_2)$, giving

$$\overline{R} = \frac{k \int_{-\theta_2}^{\theta_2} S(\theta)\, d\theta}{\sqrt{2k\theta_2}\ \sigma}$$

which is seen to be essentially equation 6. Thus it has been demonstrated that this equation applies to low-pulse-density cases, in a statistical average sense, as well as to high-pulse-density cases.

This proposition has been proved rigorously here only for the case where $2\theta_2 = (n+1)\Delta\theta$. It is also true for "integration angles" that are not integral multiples of $\Delta\theta$, but it was felt that the additional proof would not be of sufficient value to the reader to justify its inclusion.